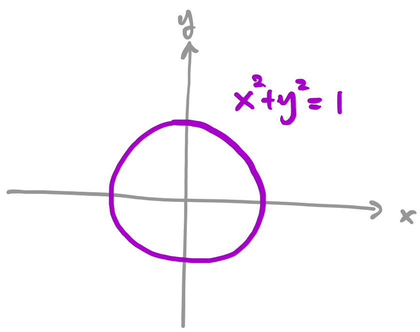
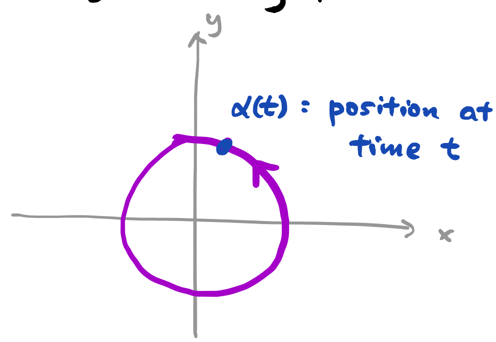
§ Curves and Arc Length

Question: What is a "curve" (in $\mathbb{R}^3$)?

There are two different ways to think of a curve:

(I) as a geometric locus OR (II) as the path described by a moving particle





Remark: We are mostly interested in the geometric shape of a curve as in (I), but (II) is more useful since we can bring in the tools from "Calculus" to describe the geometric behaviour. It is like giving a "coordinate system" along a curve which allows calculations to be done.

Definition: A (smooth parametrized) curve is a C^∞ map

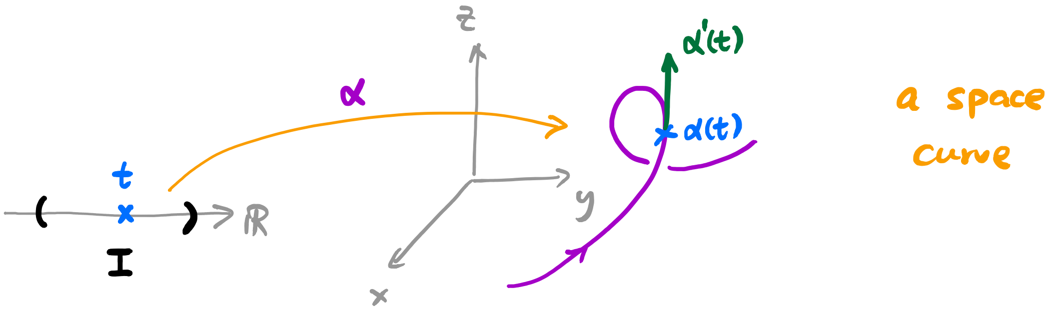
$$\alpha : I \longrightarrow \mathbb{R}^3$$

$$\alpha(t) = (x(t), y(t), z(t))$$

where $I \subset \mathbb{R}$ is a (connected) open interval, which is possibly unbounded.

tangent vector of α at $t \in I$: $\alpha'(t) = (x'(t), y'(t), z'(t))$

trace of α : $\text{image}(\alpha) = \alpha(I) \subset \mathbb{R}^3$.



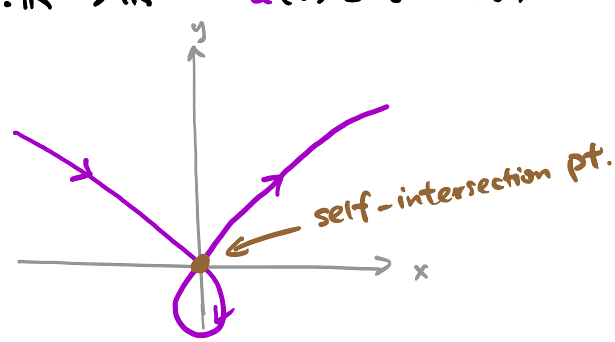
<u>Note</u>: We say that $\alpha : I \to \mathbb{R}^3$ is a plane curve if there exists a plane $P \subseteq \mathbb{R}^3$ s.t. $\alpha(I) \subseteq P$.

After a rigid motion (rotation + translation) in $\mathbb{R}^3$, we can assume that P = xy-plane, i.e.

$$\alpha(t) = (x(t), y(t), 0) \quad , \text{ i.e. } \alpha : I \to \mathbb{R}^2 .$$

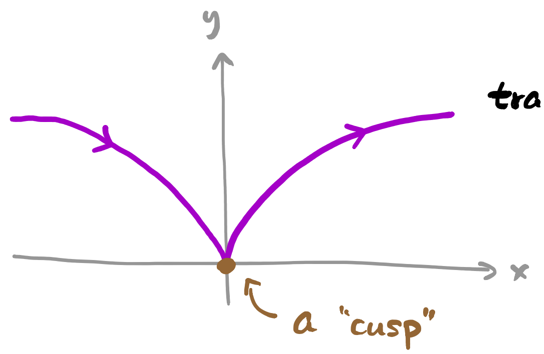
<u>Remarks</u>: 1) A curve may have self-intersections, i.e. α may not be 1-1.

<u>E.g.</u> $\alpha : \mathbb{R} \to \mathbb{R}^2 \qquad \alpha(t) = (t^3 - 4t, t^2 - 4)$



2) The trace of α may not be smooth.

E.g.: $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (t^3, t^2)$



$\text{trace}(\alpha) = \{ y = x^{2/3} \}$

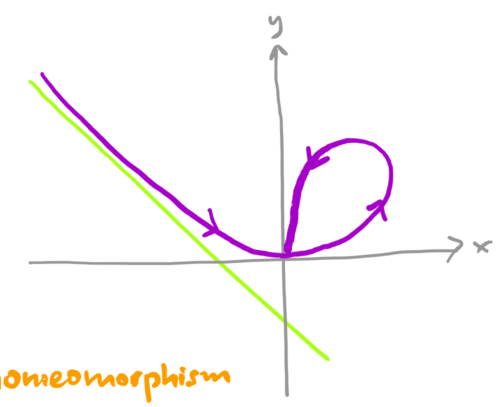
not smooth at $x = 0$!

3) Even if $\alpha : I \to \mathbb{R}^3$ is 1-1, it may not be a homeomorphism onto its image.

E.g. "Folium of Descartes"

$\alpha : (-1, \infty) \longrightarrow \mathbb{R}^2$

$$\alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right)$$



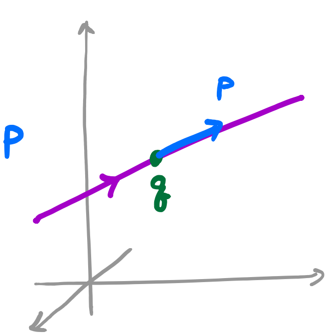
Exercise: Prove that it is not a homeomorphism onto its image.

There are three important curves which we will keep mentioning from time to time.

Example I: Straight lines

$\alpha : \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = q + t \cdot p$

is a line through q and parallel to p.

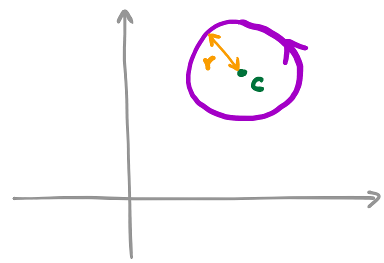


Example II : Circles

$\alpha : \mathbb{R} \to \mathbb{R}^2$

$\alpha(t) = c + r(\cos t, \sin t)$

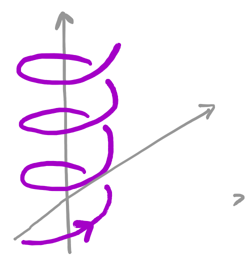
circle of radius $r > 0$ centered at c.



Example III : Helix

$\alpha : \mathbb{R} \to \mathbb{R}^3$

$\alpha(t) = (\cos t, \sin t, t)$



Definition: Let $\alpha : I \to \mathbb{R}^3$ be a curve, and $[a,b] \subset I$. The length of α from a to b is defined as

$$(*): \quad \boxed{L_a^b(\alpha) := \int_a^b |\alpha'(t)| \, dt}$$

Remark: The notion of length defined above is "geometrical" as it depends only on the geometric locus of the curve. More precisely, the length of a curve is invariant under ① rigid motions and ② reparametrization.

We now explain these two concepts.

Definition: A rigid motion $\phi : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ (as a metric space with respect to the Euclidean distance).

It is well-known that any such ϕ is an affine map of the form:

$$\boxed{\phi(x) = Ax + b} \qquad \forall\, x \in \mathbb{R}^n$$

where $A \in O(n)$, i.e. $AA^t = I = A^tA$, and $b \in \mathbb{R}^n$.

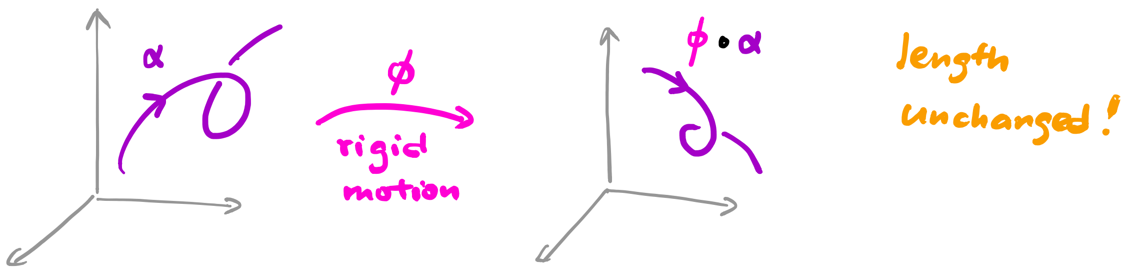
Definition: If $\det A > 0$, ϕ is orientation-preserving.
If $\det A < 0$, ϕ is orientation-reversing.

Note: $\det A = \pm 1$ if $A \in O(n)$.

Proposition: Rigid motions preserve length of curves, i.e. if $\alpha : I \to \mathbb{R}^3$ is a curve and $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid motion, then $\phi \circ \alpha : I \to \mathbb{R}^3$ is also a curve and

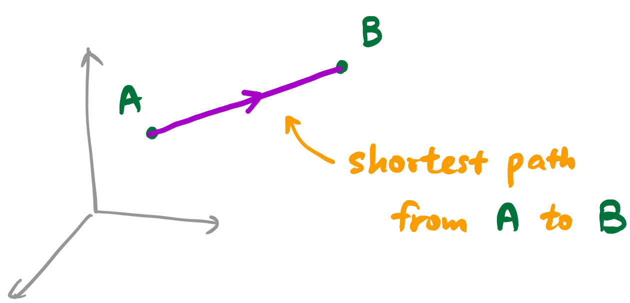
$$\boxed{L_a^b(\alpha) = L_a^b(\phi \circ \alpha)} \qquad \text{for any } [a,b] \subseteq I.$$

Proof: Exercise.



Proposition: Straight lines are the (unique) shortest curves joining two given points in $\mathbb{R}^3$.

Proof: Exercise.



Definition: A curve $\alpha : I \to \mathbb{R}^3$ is said to be parametrized by arc length (p.b.a.l.) if

$$\boxed{|\alpha'(t)| = 1 \qquad \forall\, t \in I}$$

Remark: If $\alpha : I \to \mathbb{R}^3$ is p.b.a.l., then

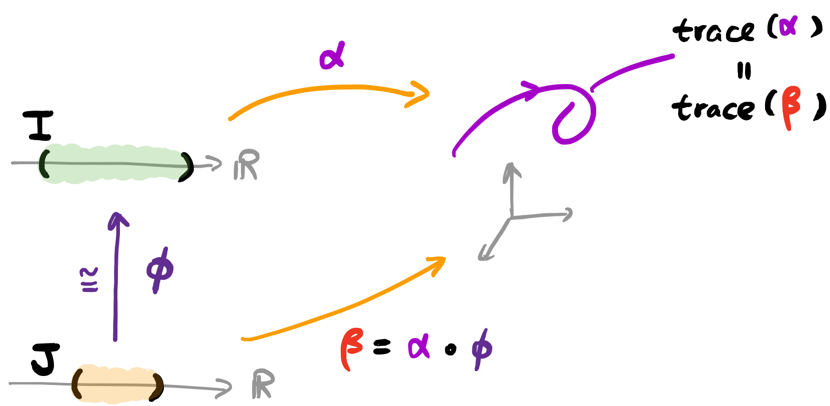
$$L_a^b(\alpha) = b - a \quad \text{for any } [a,b] \subset I.$$

Definition: Let $\alpha : I \to \mathbb{R}^3$ be a curve. For any diffeomorphism

$\phi : J \subseteq \mathbb{R} \to I$, one can define a new curve

$$\boxed{\beta = \alpha \circ \phi : J \to \mathbb{R}^3}$$

which is called a reparametrization of α.



Remark: α and β parametrize the "same" curve, i.e. their images are the same.

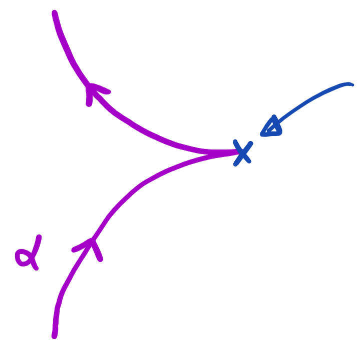
Proposition: The length of a curve is invariant under reparametrization, i.e.

$$L_a^b(\beta) = L_c^d(\alpha) \quad \text{if } \phi([a,b]) = [c,d]$$

Proof: Exercise.

§ Regular Curves

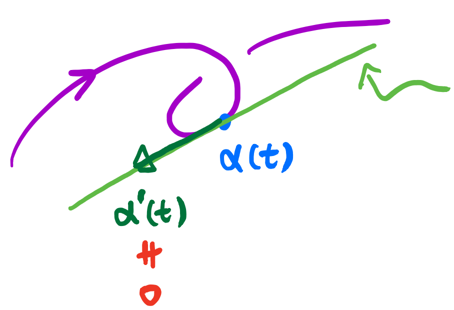
Recall: The trace of a curve may not be "smooth":



NOT "smooth" here! since $\nexists$ good approx. tangent line at this pt.

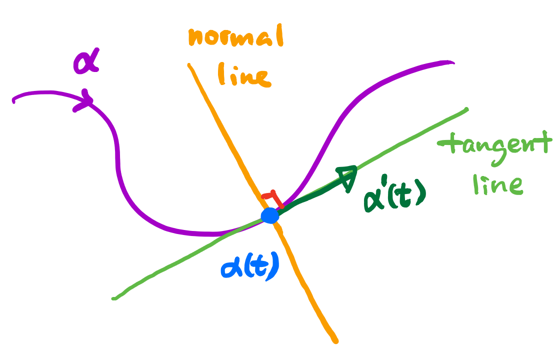
Definition: A curve $\alpha : I \to \mathbb{R}^3$ is regular if $|\alpha'(t)| \neq 0$ for all $t \in I$.

Note: For a regular curve $\alpha : I \to \mathbb{R}^3$,



tangent line at time t is well-defined!

For a regular plane curve $\alpha : I \to \mathbb{R}^3$ normal line is also defined



Q: What about for space curve?

Example: Any curve $\alpha : I \to \mathbb{R}^3$ which is p.b.a.l.
is regular. ($\because$ $|\alpha'(t)| = 1 \neq 0$)

Theorem: Any regular curve admits a reparametrization by arc length.

i.e. given a regular curve

$$\alpha : I \to \mathbb{R}^3$$

$\exists$ diffeomorphism $\phi : J \to I$ for some interval $J \subseteq \mathbb{R}$ s.t.

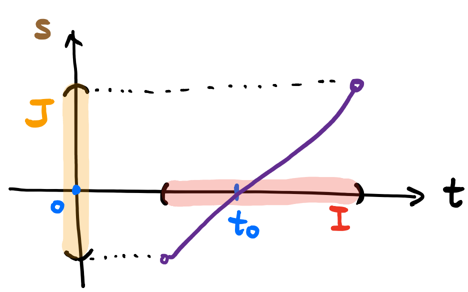
$$\beta = \alpha \circ \phi : J \to \mathbb{R}^3 \text{ is p.b.a.l.}$$

Proof: Fix any $t_0 \in I$, consider the

arc length function from t_0 : $$S(t) := \int_{t_0}^{t} |\alpha'(u)|\, du$$

Note: S differentiable with $S'(t) = |\alpha'(t)| \underset{\text{regular}}{>} 0 \quad \forall t \in I$

So, S is a continuous, strictly increasing function :



$$S : I \xrightarrow{\cong} J \subseteq \mathbb{R}$$

has a smooth inverse

$$\phi = S^{-1} : J \xrightarrow{\cong} I .$$

Define $\beta = \alpha \circ \phi : J \to \mathbb{R}^3$, which is a reparam. of α

<u>Claim</u>: β is p.b.a.l.

<u>Proof of Claim</u>:

$$\beta'(s) \underset{\substack{\uparrow \\ \text{Chain rule}}}{=} \alpha'(\phi(s)) \cdot \phi'(s) = \frac{\alpha'(\phi(s))}{S'(\phi(s))} = \underbrace{\frac{\alpha'(\phi(s))}{|\alpha'(\phi(s))|}}_{\text{unit vector!}}$$

$\square$

<u>Example</u>: $\alpha(t) = (2\cos t, 2\sin t)$, $t \in \mathbb{R}$

$$S(t) = \int_0^t |\alpha'(u)|\, du = \int_0^t 2\, du = 2t$$

inverse $\Rightarrow$ $\phi(s) = \frac{s}{2}$, $s \in \mathbb{R}$

$$\beta(s) = \alpha(\phi(s)) = \alpha\left(\tfrac{s}{2}\right) = \left(2\cos\tfrac{s}{2}, 2\sin\tfrac{s}{2}\right), \quad s \in \mathbb{R}$$

<u>Exercise</u>: Reparametrize the helix $\alpha(t) = (\cos t, \sin t, t)$ by arc length.

FROM NOW ON, we always assume a curve is p.b.a.l. unless otherwise stated.